

Quick Review 5.1

In Exercises 1–4, evaluate the expression.

1. $\sin^{-1}\left(\frac{12}{13}\right)$

2. $\cos^{-1}\left(\frac{3}{5}\right)$

3. $\cos^{-1}\left(-\frac{4}{5}\right)$

4. $\sin^{-1}\left(-\frac{5}{13}\right)$

In Exercises 5–8, factor the expression into a product of linear factors

5. $a^2 - 2ab + b^2$

6. $4u^2 + 4u + 1$

7. $2x^2 - 3xy - 2y^2$

8. $2v^2 - 5v - 3$

In Exercises 9–12, simplify the expression.

9. $\frac{1}{x} - \frac{2}{y}$

10. $\frac{a}{x} + \frac{b}{y}$

11. $\frac{x+y}{(1/x) + (1/y)}$

12. $\frac{x}{x-y} - \frac{y}{x+y}$

Section 5.1 Exercises

In Exercises 1–8, use basic identities to simplify the expression.

1. $\tan x \cos x$

2. $\cot x \tan x$

3. $\sec y \sin(\pi/2 - y)$

4. $\cot u \sin u$

5. $\frac{1 + \tan^2 x}{\csc^2 x}$

6. $\frac{1 - \cos^2 \theta}{\sin \theta}$

7. $\cos x - \cos^3 x$

8. $\frac{\sin^2 u + \tan^2 u + \cos^2 u}{\sec u}$

In Exercises 9–14, simplify the expression to either 1 or -1 .

9. $\sin x \csc(-x)$

10. $\sec(-x) \cos(-x)$

11. $\cot(-x) \cot(\pi/2 - x)$

12. $\cot(-x) \tan(-x)$

13. $\sin^2(-x) + \cos^2(-x)$

14. $\sec^2(-x) - \tan^2 x$

In Exercises 15–18, simplify the expression to either a constant or a basic trigonometric function. Support your result graphically.

15. $\frac{\tan(\pi/2 - x) \csc x}{\csc^2 x}$

16. $\frac{1 + \tan x}{1 + \cot x}$

17. $(\sec^2 x + \csc^2 x) - (\tan^2 x + \cot^2 x)$

18. $\frac{\sec^2 u - \tan^2 u}{\cos^2 v + \sin^2 v}$

In Exercises 19–24, use the basic identities to change the expression to one involving only sines and cosines. Then simplify to a basic trigonometric function.

19. $(\sin x)(\tan x + \cot x)$

20. $\sin \theta - \tan \theta \cos \theta + \cos(\pi/2 - \theta)$

21. $\sin x \cos x \tan x \sec x \csc x$

22. $\frac{(\sec y - \tan y)(\sec y + \tan y)}{\sec y}$

23. $\frac{\tan x}{\csc^2 x} + \frac{\tan x}{\sec^2 x}$

24. $\frac{\sec^2 x \csc x}{\sec^2 x + \csc^2 x}$

In Exercises 25–30, combine the fractions and simplify to a multiple of a power of a basic trigonometric function (e.g., $3 \tan^2 x$).

25. $\sin \theta + \tan \theta \cos \theta$

26. $\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x}$

27. $\frac{\sin x}{\cot^2 x} - \frac{\sin x}{\cos^2 x}$

28. $\frac{1}{\sec x - 1} - \frac{1}{\sec x + 1}$

29. $\frac{\sec x}{\sin x} - \frac{\sin x}{\cos x}$

30. $\frac{\sin x}{1 - \cos x} + \frac{1 - \cos x}{\sin x}$

In Exercises 31–38, write each expression in factored form as an algebraic expression of a single trigonometric function (e.g., $(2 \sin x + 3)(\sin x - 1)$).

31. $\cos^2 x + 2 \cos x + 1$

32. $1 - 2 \sin x + \sin^2 x$

33. $1 - 2 \sin x + (1 - \cos^2 x)$

34. $\sin x - \cos^2 x - 1$

35. $\cos x - 2 \sin^2 x + 1$

36. $\sin^2 x + \frac{2}{\csc x} + 1$

37. $4 \tan^2 x - \frac{4}{\cot x} + \sin x \csc x$

38. $\sec^2 x - \sec x + \tan^2 x$

In Exercises 39–42, write each expression as an algebraic expression of a single trigonometric function (e.g., $2 \sin x + 3$).

39. $\frac{1 - \sin^2 x}{1 + \sin x}$

40. $\frac{\tan^2 \alpha - 1}{1 + \tan \alpha}$

41. $\frac{\sin^2 x}{1 + \cos x}$

42. $\frac{\tan^2 x}{\sec x + 1}$

In Exercises 43–48, find all solutions in the interval $[0, 2\pi)$.

43. $2 \cos x \sin x - \cos x = 0$

44. $\sqrt{2} \tan x \cos x - \tan x = 0$

45. $\tan x \sin^2 x = \tan x$

46. $\sin x \tan^2 x = \sin x$

47. $\tan^2 x = 3$

48. $2 \sin^2 x = 1$

In Exercises 49–54, find all real solutions.

49. $4 \cos^2 x - 4 \cos x + 1 = 0$

50. $2 \sin^2 x + 3 \sin x + 1 = 0$

51. $\sin^2 \theta - 2 \sin \theta = 0$ 52. $3 \sin t = 2 \cos^2 t$
 53. $\cos(\sin x) = 1$ 54. $2 \sin^2 x + 3 \sin x = 2$

In Exercises 55–60, make the suggested trigonometric substitution, and then use Pythagorean Identities to write the resulting function as a multiple of a basic trigonometric function.

55. $\sqrt{1-x^2}$, $x = \cos \theta$ 56. $\sqrt{x^2+1}$, $x = \tan \theta$
 57. $\sqrt{x^2-9}$, $x = 3 \sec \theta$ 58. $\sqrt{36-x^2}$, $x = 6 \sin \theta$
 59. $\sqrt{x^2+81}$, $x = 9 \tan \theta$
 60. $\sqrt{x^2-100}$, $x = 10 \sec \theta$

Explorations

61. Write all six basic trigonometric functions entirely in terms of $\sin x$.
62. Write all six basic trigonometric functions entirely in terms of $\cos x$.
63. **Writing to Learn** Graph the functions $y = \sin^2 x$ and $y = -\cos^2 x$ in the standard trigonometric viewing window. Describe the apparent relationship between these two graphs and verify it with a trigonometric identity.
64. **Writing to Learn** Graph the functions $y = \sec^2 x$ and $y = \tan^2 x$ in the standard trigonometric viewing window. Describe the apparent relationship between these two graphs and verify it with a trigonometric identity. ■
65. **Orbit of the Moon** Because its orbit is elliptical, the distance from the moon to the earth in miles (measured from the center of the moon to the center of the earth) varies periodically. On Monday, January 11, 1999, the moon was at its apogee (farthest from the earth). The distance of the moon from the earth on the first ten Mondays of 1999 are recorded in Table 5.1:

Table 5.1 Distance from Earth to Moon

Date	Day	Distance
Jan 11	0	251,967
Jan 18	7	238,344
Jan 25	14	225,785
Feb 1	21	240,390
Feb 8	28	251,807
Feb 15	35	236,309
Feb 22	42	226,104
Mar 1	49	242,400
Mar 8	56	251,331
Mar 15	63	234,337

Source: *The World Almanac and Book of Facts, 1999.*

- (a) Draw a scatter plot of the data, using “day” as x and “distance” as y .
- (b) Use your calculator to do a sine regression of y on x . Find the equation of the best-fit sine curve and superimpose its graph on the scatter plot.

(c) What is the approximate number of days from apogee to apogee? Interpret this number in terms of the orbit of the moon.

(d) Approximately how far is the moon from the earth at perigee (closest distance)?

(e) Since the data begin at apogee, perhaps a cosine curve would be a more appropriate model. Use the sine curve in (b) and a cofunction identity to write a cosine curve that fits the data.

66. **Group Activity** Split the class into six groups. Assign each group one of the basic trigonometric functions and have each group construct a list of five different expressions that can be simplified to its function. Have each group exchange its list with its “cofunction” group to check for accuracy. Later, construct a quiz using two functions from each list, the object of the quiz being to simplify each expression to a basic trigonometric function.

Extending the Ideas

67. Prove that $\sin^4 \theta - \cos^4 \theta = \sin^2 \theta - \cos^2 \theta$.
68. Find all values of k that result in $\sin^2 x + 1 = k \sin x$ having an infinite solution set.
69. Use the cofunction identities and odd–even identities to prove that $\sin(\pi - x) = \sin x$.
 [Hint: $\sin(\pi - x) = \sin(\pi/2 - (x - \pi/2))$].
70. Use the cofunction identities and odd–even identities to prove that $\cos(\pi - x) = -\cos x$.
 [Hint: $\cos(\pi - x) = \cos(\pi/2 - (x - \pi/2))$].
71. Use the identity in Exercise 69 to prove that in any $\triangle ABC$, $\sin(A + B) = \sin C$.
72. Use the identities in Exercises 69 and 70 to find an identity for simplifying $\tan(\pi - x)$.

3. $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$

4. $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$

5. $\sin^5 x = (1 - 2 \cos^2 x + \cos^4 x)(\sin x)$

6. $\sin^2 x \cos^5 x = (\sin^2 x - 2 \sin^4 x + \sin^6 x)(\cos x)$

Example 6 PROVING AN IDENTITY USEFUL IN CALCULUS

Prove the following identity:

$$\sin^2 x \cos^5 x = (\sin^2 x - 2 \sin^4 x + \sin^6 x)(\cos x).$$

Solution We begin with the expression on the left.

$$\begin{aligned} \sin^2 x \cos^5 x &= \sin^2 x \cos^4 x \cos x \\ &= (\sin^2 x)(\cos^2 x)^2(\cos x) \\ &= (\sin^2 x)(1 - \sin^2 x)^2(\cos x) \\ &= (\sin^2 x)(1 - 2 \sin^2 x + \sin^4 x)(\cos x) \\ &= (\sin^2 x - 2 \sin^4 x + \sin^6 x)(\cos x) \end{aligned}$$

Quick Review 5.2

In Exercises 1–6, write the expression in terms of sines and cosines only. Express your answer as a single fraction.

1. $\csc x + \sec x$

2. $\tan x + \cot x$

3. $\cos x \csc x + \sin x \sec x$

4. $\sin \theta \cot \theta - \cos \theta \tan \theta$

5. $\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}$

6. $\frac{\sec \alpha}{\cos \alpha} - \frac{\sin \alpha}{\csc \alpha \cos^2 \alpha}$

In Exercises 7–12, determine whether or not the equation is an identity. If not, find a single value of x for which the two expressions are not equal.

7. $\sqrt{x^2} = x$

8. $\sqrt[3]{x^3} = x$

9. $\sqrt{1 - \cos^2 x} = \sin x$

10. $\sqrt{\sec^2 x - 1} = \tan x$

11. $\ln \frac{1}{x} = -\ln x$

12. $\ln x^2 = 2 \ln x$

Section 5.2 ExercisesIn Exercises 1–6, tell whether or not $f(x) = \sin x$ is an identity.

1. $f(x) = \frac{\sin^2 x + \cos^2 x}{\csc x}$

2. $f(x) = \frac{\tan x}{\sec x}$

3. $f(x) = \cos x \cdot \cot x$

4. $f(x) = \cos(x - \pi/2)$

5. $f(x) = (\sin^3 x)(1 + \cot^2 x)$

6. $f(x) = \frac{\sin 2x}{2}$

In Exercises 7–47, prove the identity.

7. $(\cos x)(\tan x + \sin x \cot x) = \sin x + \cos^2 x$

8. $(\sin x)(\cot x + \cos x \tan x) = \cos x + \sin^2 x$

9. $(1 - \tan x)^2 = \sec^2 x - 2 \tan x$

10. $(\cos x - \sin x)^2 = 1 - 2 \sin x \cos x$

11. $\frac{(1 - \cos u)(1 + \cos u)}{\cos^2 u} = \tan^2 u$

12. $\tan x + \sec x = \frac{\cos x}{1 - \sin x}$

13. $\frac{\cos^2 x - 1}{\cos x} = -\tan x \sin x$

14. $\frac{\sec^2 \theta - 1}{\sin \theta} = \frac{\sin \theta}{1 - \sin^2 \theta}$

15. $(1 - \sin \beta)(1 + \csc \beta) = 1 - \sin \beta + \csc \beta - \sin \beta \csc \beta$

16. $\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x} = 2 \csc^2 x$

17. $(\cos t - \sin t)^2 + (\cos t + \sin t)^2 = 2$

18. $\sin^2 \alpha - \cos^2 \alpha = 1 - 2 \cos^2 \alpha$

19. $\frac{1 + \tan^2 x}{\sin^2 x + \cos^2 x} = \sec^2 x$

20. $\frac{1}{\tan \beta} + \tan \beta = \sec \beta \csc \beta$

21. $\frac{\cos \beta}{1 + \sin \beta} = \frac{1 - \sin \beta}{\cos \beta}$ 22. $\frac{\sec x + 1}{\tan x} = \frac{\tan x}{\sec x - 1}$
23. $\frac{\tan^2 x}{\sec x + 1} = \frac{1 - \cos x}{\cos x}$ 24. $\frac{\cot v - 1}{\cot v + 1} = \frac{1 - \tan v}{1 + \tan v}$
25. $\cot^2 x - \cos^2 x = \cos^2 x \cot^2 x$
26. $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$
27. $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x$
28. $\tan^4 t + \tan^2 t = \sec^4 t - \sec^2 t$
29. $(x \sin \alpha + y \cos \alpha)^2 + (x \cos \alpha - y \sin \alpha)^2 = x^2 + y^2$
30. $\frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$ 31. $\frac{\tan x}{\sec x - 1} = \frac{\sec x + 1}{\tan x}$
32. $\frac{\sin t}{1 + \cos t} + \frac{1 + \cos t}{\sin t} = 2 \csc t$
33. $\frac{\sin x - \cos x}{\sin x + \cos x} = \frac{2 \sin^2 x - 1}{1 + 2 \sin x \cos x}$
34. $\frac{1 + \cos x}{1 - \cos x} = \frac{\sec x + 1}{\sec x - 1}$
35. $\frac{\sin t}{1 - \cos t} + \frac{1 + \cos t}{\sin t} = \frac{2(1 + \cos t)}{\sin t}$
36. $\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
37. $\sin^2 x \cos^3 x = (\sin^2 x - \sin^4 x)(\cos x)$
38. $\sin^5 x \cos^2 x = (\cos^2 x - 2 \cos^4 x + \cos^6 x)(\sin x)$
39. $\cos^5 x = (1 - 2 \sin^2 x + \sin^4 x)(\cos x)$
40. $\sin^3 x \cos^3 x = (\sin^3 x - \sin^5 x)(\cos x)$
41. $\frac{\tan x}{1 - \cot x} + \frac{\cot x}{1 - \tan x} = 1 + \sec x \csc x$
42. $\frac{\cos x}{1 + \sin x} + \frac{\cos x}{1 - \sin x} = 2 \sec x$
43. $\frac{2 \tan x}{1 - \tan^2 x} + \frac{1}{2 \cos^2 x - 1} = \frac{\cos x + \sin x}{\cos x - \sin x}$
44. $\frac{1 - 3 \cos x - 4 \cos^2 x}{\sin^2 x} = \frac{1 - 4 \cos x}{1 - \cos x}$
45. $\cos^3 x = (1 - \sin^2 x)(\cos x)$
46. $\sec^4 x = (1 + \tan^2 x)(\sec^2 x)$
47. $\sin^5 x = (1 - 2 \cos^2 x + \cos^4 x)(\sin x)$

In Exercises 48–53, match the function with one of the following. Then confirm the match with a proof. (The matching is not one-to-one.)

- (a) $\sec^2 x \csc^2 x$ (b) $\sec x + \tan x$ (c) $2 \sec^2 x$
- (d) $\tan x \sin x$ (e) $\sin x \cos x$
48. $\frac{1 + \sin x}{\cos x}$ 49. $(1 + \sec x)(1 - \cos x)$
50. $\sec^2 x + \csc^2 x$ 51. $\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x}$
52. $\frac{1}{\tan x + \cot x}$ 53. $\frac{1}{\sec x - \tan x}$

Explorations

In Exercises 54–59, identify a simple function that has the same graph. Then confirm your choice with a proof.

54. $\sin x \cot x$ 55. $\cos x \tan x$
56. $\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}$ 57. $\frac{\csc x}{\sin x} - \frac{\cot x \csc x}{\sec x}$
58. $\frac{\sin x}{\tan x}$ 59. $(\sec^2 x)(1 - \sin^2 x)$ ■

60. Writing to Learn Let θ be any number that is in the domain of all six trig functions. Explain why the natural logarithms of all six basic trig functions of θ sum to 0.

61. If A and B are complementary angles, prove that $\sin^2 A + \sin^2 B = 1$.

62. Group Activity If your class contains $2n$ students, write the two expressions from n different identities on separate pieces of paper. (You can use identities from Exercises 1–47 of this section or from other textbooks, but be sure to write them all in the same variable.) Mix up the slips of paper and give one to each student. Then see how long it takes the class, without looking at the book, to pair themselves off as identities. (This activity takes on an added degree of difficulty if graphers are not allowed.)

If the class contains an odd number of students, the teacher can join in to make it even.

Extending the Ideas

In Exercises 63–68, confirm the identity.

63. $\sqrt{\frac{1 - \sin t}{1 + \sin t}} = \frac{1 - \sin t}{|\cos t|}$ 64. $\sqrt{\frac{1 + \cos t}{1 - \cos t}} = \frac{1 + \cos t}{|\sin t|}$
65. $\sin^6 x + \cos^6 x = 1 - 3 \sin^2 x \cos^2 x$
66. $\cos^6 x - \sin^6 x = (\cos^2 x - \sin^2 x)(1 - \cos^2 x \sin^2 x)$
67. $\ln |\tan x| = \ln |\sin x| - \ln |\cos x|$
68. $\ln |\sec \theta + \tan \theta| + \ln |\sec \theta - \tan \theta| = 0$

69. Writing to Learn Let

$$y_1 = [\sin(x + 0.001) - \sin x]/0.001 \text{ and } y_2 = \cos x.$$

(a) Use graphs and tables to decide whether $y_1 = y_2$.

(b) Find a value for h so that the graph of $y_3 = y_1 - y_2$ in $[-2\pi, 2\pi]$ by $[-h, h]$ appears to be a sinusoid. Give a convincing argument that y_3 is a sinusoid.

70. Hyperbolic Functions The *hyperbolic trigonometric functions* are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \quad \operatorname{sech} x = \frac{1}{\cosh x} \quad \operatorname{coth} x = \frac{1}{\tanh x}$$

Confirm the identity.

- (a) $\cosh^2 x - \sinh^2 x = 1$
- (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$
- (c) $\operatorname{coth}^2 x - 1 = \operatorname{csch}^2 x$

Quick Review 5.3

In Exercises 1–6, express the angle as a sum or difference of special angles (multiples of 30° , 45° , $\pi/6$, or $\pi/4$).

- 15°
- 75°
- 165°
- $\pi/12$
- $5\pi/12$
- $7\pi/12$

In Exercises 7–10, tell whether or not the identity $f(x + y) = f(x) + f(y)$ holds for the function f .

- $f(x) = \ln x$
- $f(x) = e^x$
- $f(x) = 32x$
- $f(x) = x + 10$

Section 5.3 Exercises

In Exercises 1–10, use a sum or difference identity to find an exact value.

- $\sin 15^\circ$
- $\tan 15^\circ$
- $\sin 75^\circ$
- $\cos 75^\circ$
- $\cos \frac{\pi}{12}$
- $\sin \frac{7\pi}{12}$
- $\tan \frac{5\pi}{12}$
- $\tan \frac{11\pi}{12}$
- $\cos \frac{7\pi}{12}$
- $\sin \frac{-\pi}{12}$

In Exercises 11–22, write the expression as the sine, cosine, or tangent of an angle.

- $\sin 42^\circ \cos 17^\circ - \cos 42^\circ \sin 17^\circ$
- $\cos 94^\circ \cos 18^\circ + \sin 94^\circ \sin 18^\circ$
- $\sin \frac{\pi}{5} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cos \frac{\pi}{5}$
- $\sin \frac{\pi}{3} \cos \frac{\pi}{7} - \sin \frac{\pi}{7} \cos \frac{\pi}{3}$
- $\frac{\tan 19^\circ + \tan 47^\circ}{1 - \tan 19^\circ \tan 47^\circ}$
- $\frac{\tan(\pi/5) - \tan(\pi/3)}{1 + \tan(\pi/5) \tan(\pi/3)}$
- $\cos \frac{\pi}{7} \cos x + \sin \frac{\pi}{7} \sin x$
- $\cos x \cos \frac{\pi}{7} - \sin x \sin \frac{\pi}{7}$

19. $\sin 3x \cos x - \cos 3x \sin x$

20. $\cos 7y \cos 3y - \sin 7y \sin 3y$

21. $\frac{\tan 2y + \tan 3x}{1 - \tan 2y \tan 3x}$

22. $\frac{\tan 3\alpha - \tan 2\beta}{1 + \tan 3\alpha \tan 2\beta}$

In Exercises 23–30, prove the identity.

23. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

24. $\tan\left(x - \frac{\pi}{2}\right) = -\cot x$

25. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

26. $\cos\left[\left(\frac{\pi}{2} - x\right) - y\right] = \sin(x + y)$

27. $\sin\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x$

28. $\cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (\cos x + \sin x)$

29. $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1 + \tan \theta}{1 - \tan \theta}$

30. $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin \theta$

In Exercises 31–34, match each graph with a pair of the following equations. Use your knowledge of identities and transformations, not your grapher.

(a) $y = \cos(3 - 2x)$

(b) $y = \sin x \cos 1 + \cos x \sin 1$

(c) $y = \cos(x - 3)$

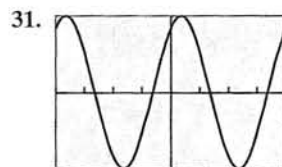
(d) $y = \sin(2x - 5)$

(e) $y = \cos x \cos 3 + \sin x \sin 3$

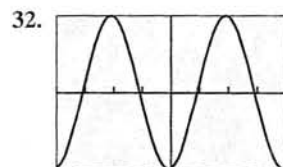
(f) $y = \sin(x + 1)$

(g) $y = \cos 3 \cos 2x + \sin 3 \sin 2x$

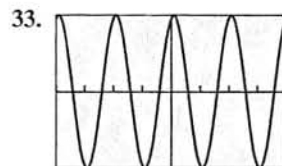
(h) $y = \sin 2x \cos 5 - \cos 2x \sin 5$



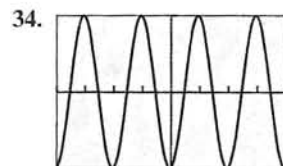
$[-2\pi, 2\pi]$ by $[-1, 1]$



$[-2\pi, 2\pi]$ by $[-1, 1]$



$[-2\pi, 2\pi]$ by $[-1, 1]$



$[-2\pi, 2\pi]$ by $[-1, 1]$

In Exercises 35 and 36, use sum or difference identities (and not your grapher) to solve the equation exactly.

35. $\sin 2x \cos x = \cos 2x \sin x$

36. $\cos 3x \cos x = \sin 3x \sin x$

In Exercises 37–42, prove the reduction formula.

37. $\sin\left(\frac{\pi}{2} - u\right) = \cos u$

38. $\tan\left(\frac{\pi}{2} - u\right) = \cot u$

39. $\cot\left(\frac{\pi}{2} - u\right) = \tan u$

40. $\sec\left(\frac{\pi}{2} - u\right) = \csc u$

41. $\csc\left(\frac{\pi}{2} - u\right) = \sec u$

42. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

In Exercises 43–46, express the function as a sinusoid in the form $y = a \sin(bx + c)$.

43. $y = 3 \sin x + 4 \cos x$

44. $y = 5 \sin x - 12 \cos x$

45. $y = \cos 3x + 2 \sin 3x$

46. $y = 3 \cos 2x - 2 \sin 2x$

In Exercises 47–55, prove the identity.

47. $\sin(x - y) + \sin(x + y) = 2 \sin x \cos y$

48. $\cos(x - y) + \cos(x + y) = 2 \cos x \cos y$

49. $\cos 3x = \cos^3 x - 3 \sin^2 x \cos x$

50. $\sin 3u = 3 \cos^2 u \sin u - \sin^3 u$

51. $\cos 3x + \cos x = 2 \cos 2x \cos x$

52. $\sin 4x + \sin 2x = 2 \sin 3x \cos x$

53. $\tan(x + y) \tan(x - y) = \frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}$

54. $\tan 5u \tan 3u = \frac{\tan^2 4u - \tan^2 u}{1 - \tan^2 4u \tan^2 u}$

55. $\frac{\sin(x + y)}{\sin(x - y)} = \frac{(\tan x + \tan y)}{(\tan x - \tan y)}$

Explorations

56. Prove the identity $\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$.

57. Prove the identity $\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$.

58. **Writing to Learn** Explain why the identity in Exercise 56 cannot be used to prove the reduction formula $\tan(x + \pi/2) = -\cot x$. Then prove the reduction formula.

59. **Writing to Learn** Explain why the identity in Exercise 57 cannot be used to prove the reduction formula $\tan(x - 3\pi/2) = -\cot x$. Then prove the reduction formula.

60. **An Identity for Calculus** Prove the following identity, which is used in calculus to prove an important differentiation formula.

$$\frac{\sin(x + h) - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \frac{\sin h}{h}$$

61. **An Identity for Calculus** Prove the following identity, which is used in calculus to prove another important differentiation formula.

$$\frac{\cos(x + h) - \cos x}{h} = \cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \frac{\sin h}{h}$$

62. **Group Activity** Place 24 points evenly spaced around the unit circle, starting with the point (1, 0). Using only your knowledge of the special angles and the sum and difference identities, work with your group to find the exact coordinates of all 24 points.

Extending the Ideas

In Exercises 63–66, assume that A , B , and C are the three angles of some $\triangle ABC$. (Note, then, that $A + B + C = \pi$.) Prove the following identities.

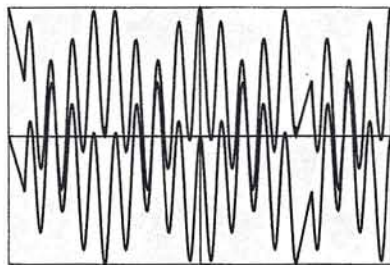
63. $\sin(A + B) = \sin C$

64. $\cos C = \sin A \sin B - \cos A \cos B$

65. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

66. $\cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C = -1$

67. **Writing to Learn** The figure shows graphs of $y_1 = \cos 5x \cos 4x$ and $y_2 = -\sin 5x \sin 4x$ in one viewing window. Discuss the question, “How many solutions are there to the equation $\cos 5x \cos 4x = -\sin 5x \sin 4x$ in the interval $[-2\pi, 2\pi]$?” Give an algebraic argument that answers the question more convincingly than the graph does. Then support your argument with an appropriate graph.



$[-2\pi, 2\pi]$ by $[-1, 1]$

68. **Harmonic Motion** Alternating electric current, an oscillating spring, or any other harmonic oscillator can be modeled by the equation

$$x = a \cos\left(\frac{2\pi}{T}t + \delta\right),$$

where T is the time for one period and δ is the phase constant. Show that this motion can also be modeled by the following sum of cosine and sine, each with zero phase constant:

$$a_1 \cos\left(\frac{2\pi}{T}t\right) + a_2 \sin\left(\frac{2\pi}{T}t\right),$$

where $a_1 = a \cos \delta$ and $a_2 = -a \sin \delta$.

69. **Magnetic Fields** A magnetic field B can sometimes be modeled as the sum of an incident and a reflective field as

$$B = B_{\text{in}} + B_{\text{ref}},$$

where

$$B_{\text{in}} = \frac{E_0}{c} \cos\left(\omega t - \frac{\omega x}{c}\right), \quad \text{and}$$

$$B_{\text{ref}} = \frac{E_0}{c} \cos\left(\omega t + \frac{\omega x}{c}\right).$$

Show that

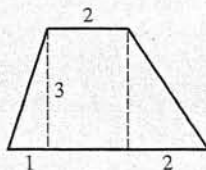
$$B = 2 \frac{E_0}{c} \cos \omega t \cos \frac{\omega x}{c}.$$

Quick Review 5.4

In Exercises 1–8, find the general solution of the equation.

- $\tan x - 1 = 0$
- $\tan x + 1 = 0$
- $(\cos x)(1 - \sin x) = 0$
- $(\sin x)(1 + \cos x) = 0$
- $\sin x + \cos x = 0$
- $\sin x - \cos x = 0$
- $(2 \sin x - 1)(2 \cos x + 1) = 0$
- $(\sin x + 1)(2 \cos x - \sqrt{2}) = 0$

9. Find the area of the trapezoid.



10. Find the height of the isosceles triangle.



Section 5.4 Exercises

In Exercises 1–4, use the appropriate sum or difference identity to prove the double-angle identity.

- $\cos 2u = \cos^2 u - \sin^2 u$
- $\cos 2u = 2 \cos^2 u - 1$
- $\cos 2u = 1 - 2 \sin^2 u$
- $\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$

In Exercises 5–10, find all solutions to the equation in the interval $[0, 2\pi)$.

- $\sin 2x = 2 \sin x$
- $\sin 2x = \sin x$
- $\cos 2x = \sin x$
- $\cos 2x = \cos x$
- $\sin 2x - \tan x = 0$
- $\cos^2 x + \cos x = \cos 2x$

In Exercises 11–14, write the expression as one involving only $\sin \theta$ and $\cos \theta$.

- $\sin 2\theta + \cos \theta$
- $\sin 2\theta + \cos 2\theta$
- $\sin 2\theta + \cos 3\theta$
- $\sin 3\theta + \cos 2\theta$

In Exercises 15–22, prove the identity.

- $\sin 4x = 2 \sin 2x \cos 2x$
- $\cos 6x = 2 \cos^2 3x - 1$
- $2 \csc 2x = \csc^2 x \tan x$
- $2 \cot 2x = \cot x - \tan x$
- $\sin 3x = (\sin x)(4 \cos^2 x - 1)$
- $\sin 3x = (\sin x)(3 - 4 \sin^2 x)$
- $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$
- $\sin 4x = (4 \sin x \cos x)(2 \cos^2 x - 1)$

In Exercises 23–30, find all solutions in the interval $[0, 2\pi)$. Solve algebraically whenever possible. (Only Exercises 24, 29, and 30 should require a grapher.)

- $\cos 2x + \cos x = 0$
- $\cos 2x + 2 \sin x = 0$
- $\cos x + \cos 3x = 0$
- $\sin x + \sin 3x = 0$
- $\sin 2x + \sin 4x = 0$
- $\cos 2x + \cos 4x = 0$
- $\sin 2x - \cos 3x = 0$
- $\sin 3x + \cos 2x = 0$

In Exercises 31–36, use half-angle identities to find an exact value without a calculator.

- $\sin 15^\circ$
- $\tan 195^\circ$

33. $\cos 75^\circ$

34. $\sin 5\pi/12$

35. $\tan 7\pi/12$

36. $\cos \pi/8$

37. Prove the power-reducing identities:

(a) $\sin^2 u = \frac{1 - \cos 2u}{2}$ (b) $\cos^2 u = \frac{1 + \cos 2u}{2}$

38. (a) Use the identities in Exercise 37 to prove the power-reducing identity $\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$.

(b) **Writing to Learn** Explain why the identity in (a) does not imply that $\tan u = \sqrt{\frac{1 - \cos 2u}{1 + \cos 2u}}$.

In Exercises 39–42, use the power-reducing identities to prove the identity.

39. $\sin^4 x = \frac{1}{8}(3 - 4 \cos 2x + \cos 4x)$

40. $\cos^3 x = \left(\frac{1}{2} \cos x\right)(1 + \cos 2x)$

41. $\sin^3 2x = \left(\frac{1}{2} \sin 2x\right)(1 - \cos 4x)$

42. $\sin^5 x = \left(\frac{1}{8} \sin x\right)(3 - 4 \cos 2x + \cos 4x)$

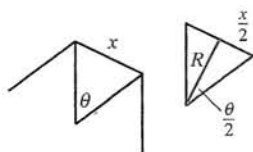
In Exercises 43–46, use the half-angle identities to find all solutions in the interval $[0, 2\pi)$. Then find the general solution.

43. $\cos^2 x = \sin^2\left(\frac{x}{2}\right)$ 44. $\sin^2 x = \cos^2\left(\frac{x}{2}\right)$

45. $\tan\left(\frac{x}{2}\right) = \frac{1 - \cos x}{1 + \cos x}$ 46. $\sin^2\left(\frac{x}{2}\right) = 2 \cos^2 x - 1$

Explorations

- 47. Connecting Trigonometry and Geometry** In a regular polygon all sides are the same length and all angles are equal in measure.



Regular polygon
with n sides

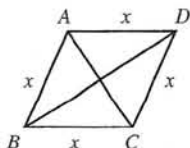
- (a) If the perpendicular distance from the center of the polygon with n sides to the midpoint of a side is R , and if the length of the side of the polygon is x , show that

$$x = 2R \tan \frac{\theta}{2}$$

where $\theta = 2\pi/n$ is the central angle subtended by one side.

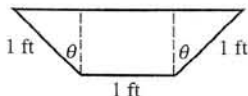
- (b) If the length of one side of a regular 11-sided polygon is approximately 5.87 and R is a whole number, what is the value of R ?

- 48. Connecting Trigonometry and Geometry** A rhombus is a quadrilateral with equal sides. The diagonals of a rhombus bisect the angles of the rhombus and are perpendicular bisectors of each other. Let $\angle ABC = \theta$, $d_1 =$ length of AC , and $d_2 =$ length of BD .

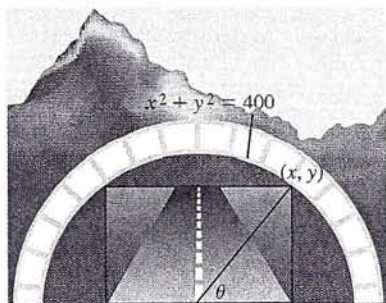


- (a) Show that $\cos \frac{\theta}{2} = \frac{d_2}{2x}$ and $\sin \frac{\theta}{2} = \frac{d_1}{2x}$.
- (b) Show that $\sin \theta = \frac{d_1 d_2}{2x^2}$.

- 49. Group Activity Maximizing Volume** The ends of a 10-foot-long water trough are isosceles trapezoids as shown in the figure. Find the value of θ that maximizes the volume of the trough and the maximum volume.



- 50. Group Activity Tunnel Problem** A rectangular tunnel is cut through a mountain to make a road. The upper vertices of the rectangle are on the circle $x^2 + y^2 = 400$, as illustrated in the figure.



- (a) Show that the cross-sectional area of the end of the tunnel is $400 \sin 2\theta$.

- (b) Find the dimensions of the rectangular end of the tunnel that maximizes its cross-sectional area.

Extending the Ideas

In Exercises 51–55, prove the double-angle formulas.

51. $\csc 2u = \frac{1}{2} \csc u \sec u$ 52. $\cot 2u = \frac{\cot^2 u - 1}{2 \cot u}$
53. $\sec 2u = \frac{\csc^2 u}{\csc^2 u - 2}$ 54. $\sec 2u = \frac{\sec^2 u}{2 - \sec^2 u}$
55. $\sec 2u = \frac{\sec^2 u \csc^2 u}{\csc^2 u - \sec^2 u}$

- 56. Writing to Learn** Explain why

$$\sqrt{\frac{1 - \cos 2x}{2}} = |\sin x|$$

is an identity but

$$\sqrt{\frac{1 - \cos 2x}{2}} = \sin x$$

is not an identity.

- 57. Sahara Sunset** Table 5.2 gives the time of day that astronomical twilight began in northeastern Mali on the first day of each month of 1999.



Table 5.2 Astronomical Twilight

Date	Day	Time	6:00 +
Jan 1	1	6:51	51
Feb 1	32	7:08	68
Mar 1	60	7:18	78
Apr 1	91	7:28	88
May 1	121	7:42	102
Jun 1	152	7:59	119
Jul 1	182	8:07	127
Aug 1	213	7:56	116
Sep 1	244	7:30	90
Oct 1	274	7:01	61
Nov 1	305	6:40	40
Dec 1	335	6:37	37